ON THE STABILITY OF A NONLINEAR SYSTEM OF AUTOMATIC CONTROL

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Considered is an automatic control system with nonlinear characteristics of the control object and with an essentially nonlinear characteristic of the controlling element.

With the aid of Liapunov's method [1] an investigation is made of the stability of the undisturbed motion of the system for the case when the characteristic equation of the system has two zero roots, and when all its other roots have negative real parts. Use is made of certain results obtained earlier by Kamenkov [2,3].

1. The equations of the disturbed motion of the system are assumed to have the form

$$\frac{dx_k}{dt} = \sum_{\alpha=1}^{n+1} b_{k\alpha} x_\alpha + n_k x_{n+2} \qquad (k = 1, \ldots, n+1)$$

$$\frac{dx_{n+2}}{dt} = f(\sigma), \qquad \sigma = \sum_{\alpha=1}^{n+1} p_\alpha x_\alpha + p_{n+2} x_{n+2} \qquad (1.1)$$

Here the x_k are generalized coordinates of the control object; x_{n+2} is the coordinate of the controlling element; σ is the control signal, and $b_{k\alpha}$, n_k, p_α, p_{n+2} are known constant parameters.

We shall assume that $f(\sigma)$ can be approximated by a function of the form

$$f(\sigma) = K\sigma^N + K_1\sigma^{N+1} + \ldots \qquad (N > 2) \qquad (1.2)$$

Let us suppose that the characteristic equation of the object of control has one zero root and n roots with negative real parts. This corresponds to the neutrality of the object with respect to one of its $n + 1$

coordinates and stability with respect to the n remaining coordinates. Therefore, the characteristic equation of the entire system will have two zero roots; for example, let $\lambda_{n+1} = \lambda_{n+2} = 0$. The remaining roots $\lambda_1, \ldots, \lambda_n$ will have negative real parts.

The problem considered in this note is the determination of the conditions of stability of the undisturbed motion of the system (1.1) under the above-made assumptions.

2. Let us assume that the roots of the characteristic equation of the system (1.1) are known. We reduce the system (1.1) to the canonical form of Lur'e [4]

$$\frac{dz_s}{dt} = \lambda_s z_s + f(\sigma) \qquad (s = 1, \ldots, n), \qquad \frac{dz_{n+1}}{dt} = f(\sigma)$$

$$\frac{d\sigma}{dt} = \sum_1^n \beta_s z_s + \beta_{n+1} z_{n+1} - rf(\sigma) \tag{2.1}$$

Here, $\lambda_1, \ldots, \lambda_n$ are nonzero roots of the characteristic equation of the object of control. The parameters of the transformation

$$z_s = \sum_{\alpha=1}^{n+1} C_{s\alpha} x_\alpha + x_{n+2} \qquad (s = 1, \ldots, n+1) \tag{2.2}$$

and also the quantities $\beta_1, \ldots, \beta_{n+1}$ are determined by the method given in [4].

It is obvious that the characteristic equation of the system (2.1) has two zero roots, while the remaining n roots have by hypothesis negative real parts.

Let us set $\sigma = \sigma_1 + A_1 z_1 + \ldots + A_n z_n$. Then

$$\frac{d\sigma_1}{dt} = \beta_{n+1} z_{n+1} + \sum_{\alpha=1}^n (\beta_\alpha - A_\alpha \lambda_\alpha) z_\alpha - f(\sigma)\Big[\sum_{\alpha=1}^n A_\alpha + r\Big]$$

Next we set $A_\alpha = \beta_\alpha/\lambda_\alpha$ and introduce the notation $\sigma_1 = x$:

$$\beta_{n+1} z_{n+1} - Bf(\sigma) = y, \qquad A_1 + \ldots + A_n + r = B$$

The system (2.1) can then be represented in the form

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = Y(x, y, z_1, \ldots, z_n)$$

$$\frac{dz_s}{dt} = \lambda_s z_s + X_s(x, y, z_1, \ldots, z_n) \qquad (s = 1, \ldots, n) \tag{2.3}$$

Here

$$\sigma = \sigma(x, y, z_1, \ldots, z_n), \qquad X_s = f(\sigma)$$

$$Y = \beta_{n+1} f(\sigma) - BKN\sigma^{N-1}\left\{y + \sum_{\alpha=1}^{n} A_\alpha [\lambda_\alpha z_\alpha + f(\sigma)\right\} + \ldots$$

The expansions of the functions Y and X_s in power series will not contain any terms of degree less than two.

Following [2], we can write the functions Y and X_s in the form

$$Y(x, y, z_1, \ldots, z_n) = Y_0(x, y) + Y_1(x, y, z_1, \ldots, z_n)$$
$$X_s(x, y, z_1, \ldots, z_n) = X_{s0}(x, y) + X_{s1}(x, y, z_1, \ldots, z_n) \qquad (2.4)$$

Here

$$Y_1(x, y, 0, \ldots, 0) = 0, \quad X_{s1}(x, y, 0, \ldots, 0) = 0$$

and the functions $Y_0(x, y)$ and $X_{s0}(x, y)$, in accordance with [2], are expressed in the form

$$Y_0(x, y) = f_0(x) + y\varphi_0(x) + y^2\psi_0(x) + \ldots \qquad (2.5)$$
$$X_{s0}(x, y) = f_s(x) + y\varphi_s(x) + y^2\psi_s(x) + \ldots$$

Here

$$f_0(x) = a_0 x^{\alpha_0} + \ldots, \qquad \varphi_0(x) = b_0 x^{\beta_0} + \ldots$$
$$f_s(x) = a_{s0} x^{\alpha_s} + \ldots, \qquad \varphi_s(x) = b_{s0} x^{\beta_s} + \ldots .$$

We thus obtain the following equations for the system under consideration:

$$f_s(x) = Kx^N, \ \varphi_s(x) = \psi_s(x) = \ldots = 0, f_0(x) = \beta_{n+1} K x^N - BK^2 N \sum_{s=1}^{n} A_s x^{2N-1},$$
$$\varphi_0(x) = -BKNx^{N-1}, \ \psi_0(x) = \ldots = 0$$

Therefore

$$a_0 = \beta_{n+1} K, \quad \alpha_0 = N, \quad b_0 = -BKN, \beta_0 = N - 1$$

Next, in order to simplify the construction of the Liapunov-Chetaev function, we introduce the transformation

$$z_s = y_s + u_s(x) + y v_s(x) \qquad (2.6)$$

The functions $u_s(x)$ and $v_s(x)$ have to be determined in such a way that the degree of the lowest-degree term in the expansion of the function $f_s(x)$ be not higher than the degree of the lowest term of the expansion of $f_0(x)$.

By retaining the notion (2.5) in the transformed system, we obtain the following equations for the determination of the functions $u_s(x)$ and $v_s(x)$:

$$\lambda_s u_s(x) + f_s(x) + X_{s1}(x, 0, u_s) = 0 \tag{2.7}$$

$$v_s(x)\varphi_0(x) + \varphi_s(x) = 0 \tag{2.8}$$

It is important to note that the lowest-degree term in x of the function $u_s(x)$ and $v_s(x)$ are equal to N and $N-1$, respectively. For such a choice of the functions $u_s(x)$ and $v_s(x)$, the validity of the equations $f_s(x) = -v_s(x)f_0(x)$, and $\phi_s(x) = -v_s(x)\phi_0(x)$ will be guaranteed, and hence the made assumptions will also be fulfilled. The quantities a_0, b_0, α_0, β_0 will hereby not be changed, but in the transformed system we have

$$\alpha_s \geqslant \alpha_0 + N - 1, \qquad \beta_{sk} \geqslant \alpha_0 + N - 2 \tag{2.9}$$

We can represent the transformed system in the form

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = f_0(x) + y\varphi_0(x) + \ldots + \Big\{\sum_{k_1+k_2=0}^{\infty} x^{k_1}y^{k_2}P^{(k_1,k_2)}(y_1, \ldots, y_n)\Big\}$$

$$\frac{dy_s}{dt} = \lambda_s y_s + f_s(x) + \sum_{k=1}^{\infty} y^k \varphi_{sk}(x) + \sum_{k_1+k_2=0}^{\infty} x^{k_1}y^{k_2}P_s^{(k_1,k_2)}(y_1, \ldots, y_n)$$

$$(s = 1, \ldots, n) \tag{2.10}$$

The conditions (2.9) exclude the effect of the nonlinear terms of the right-hand side of the third equation of (2.10) on the stability of the system. It remains to transform the system (2.10) in such a way as to exclude the effect on the stability of the terms contained within the braces of the second equation of (2.10). It was shown in [3], that such a transformation, which does not change the stability problem, does exist and can be found. The summation on the right-hand side of the second equation of (2.10) can thus start with the indices k_1 and k_2 satisfying the condition $k_1 + k_2 \geqslant a_0 + N - 1$. This transformation does not change the first $a_0 + N - 1$ terms of the expansion of the function $f_0(x)$ in powers of x, and the first $a_0 + N - 2$ terms of the expansion of the function $\phi_0(x)$ in powers of x. Thus, having established the existence of such a transformation it may still be impossible to perform this transformation because the a_0, b_0, α_0, and β_0 may not change.

After the performed transformations, the criterion of stability can be obtained by the method of [2]. For the stability of the undisturbed motion of the system (1.1) it is necessary and sufficient that the following conditions be fulfilled for odd N:

$$\beta_{n+1}K < 0, \qquad BKN > 0 \tag{2.11}$$

The requirement that N be odd can be reduced to the requirement that the characteristic of the controlling element be odd. The condition (2.11) makes it possible to construct the region of admissible values of the parameters of the control system by starting with the stability of the undisturbed motion of the system (1.1).

3. Let us consider the case when the roots of the characteristic equation of the system (1.1) are not known. One can determine the indicated properties of the roots of the characteristic equation of the system in this case directly on the basis of the coefficients of the equation by the Hurwitz-Routh criterion without solving the equation.

Let us consider Equation (1.1). We introduce the following transformations:

$$x = \sum_{k=1}^{n+2} A_k x_k, \qquad y = \sum_{k=1}^{n+2} B_k x_k \tag{3.1}$$

The coefficients A_k and B_k are determined by means of the equations

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = Y(x_1, \ldots, x_{n+2})$$

Here $Y(x_1, \ldots, x_{n+2})$ is a holomorphic function of its variables which does not contain any terms of degree less than two.

For the determination of A_k and B_k we obtain identities which give the required number of equations

$$\begin{gathered}\sum_{k=1}^{n+2} A_k \Big(\sum_{\alpha=1}^{n+1} b_{k\alpha} x_\alpha + n_k x_{n+2}\Big) \equiv \sum_{k=1}^{n+2} B_k x_k \\ \sum_{k=1}^{n+2} B_k \Big(\sum_{\alpha=1}^{n+1} b_{k\alpha} x_\alpha + n_k x_{n+2}\Big) \equiv 0\end{gathered} \tag{3.2}$$

Finding A_k and B_k, one can, by means of (3.1), express x_{n+1} and x_{n+2} in terms of x, y, $x_1, \ldots, x_n$ and substitute them in the system (1.1). Then the system (1.1) will take the form

$$\begin{gathered}\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = Y(x, y, x_1, \ldots, x_n) \\ \frac{dx_s}{dt} = \sum_{\alpha=1}^{n} q_{s\alpha} x_\alpha + p_s x + q_s y \qquad (s = 1, \ldots, n)\end{gathered} \tag{3.3}$$

Here

$$\sigma = \sum_{\alpha=1}^{n} p_\alpha' x_\alpha + p_x x + p_y y, \qquad Y = B_{n+2} f(\sigma) + A_{n+2} \frac{df(\sigma)}{dt}$$

Introducing the transformation $x_s = y_s + C_s x + D_s y$, and determining C_s and D_s by means of the equations

$$\sum_{\alpha=1}^{n} q_{s\alpha} C_\alpha + p_s = 0, \qquad \sum_{\alpha=1}^{n} q_{s\alpha} D_\alpha + q_s - C_s = 0 \tag{3.4}$$

we obtain in place of (3.3) the following system:

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = Y(x, y, y_1, \ldots, y_n)$$

$$\frac{dy_s}{dt} = \sum_{\alpha=1}^{n} q_{s\alpha} y_\alpha + X_s(x, y, y_1, \ldots, y_n) \qquad (s = 1, \ldots, n) \tag{3.5}$$

Here

$$\sigma = \sum_{\alpha=1}^{n} p_\alpha' x_\alpha + \Big(\sum_{\alpha=1}^{n} p_\alpha' C_\alpha + p_x\Big) x + \Big(\sum_{\alpha=1}^{n} p_\alpha' D_\alpha + p_y\Big) y$$

$$Y = B_{n+2} f(\sigma) + A_{n+2} K N \sigma^{N-1} \Big\{\sum_{\alpha=1}^{n} p_\alpha' \Big(\sum_{\beta=1}^{n} q_{\alpha\beta} y_\beta\Big) -$$

$$- \sum_{\alpha=1}^{n} p_\alpha' D_\alpha \Big[B_{n+2} f(\sigma) + A_{n+2} \frac{df(\sigma)}{dt}\Big] + \Big(\sum_{\alpha=1}^{n} p_\alpha' C_\alpha + p_x\Big) y +$$

$$+ \Big(\sum_{\alpha=1}^{n} p_\alpha' D_\alpha + p_y\Big)\Big[B_{n+2} f(\sigma) + A_{n+2} \frac{df(\sigma)}{dt}\Big]\Big\} \qquad X_s = - D_s Y$$

We retain the representations (2.4) and (2.5) for the functions Y and X_s. Expanding the expression for Y, we find the following lowest-degree terms of the expansions of the functions $f_0(x)$ and $\phi_0(x)$:

$$\alpha_0 = N, \qquad \beta_0 = N - 1, \qquad \alpha_s = N, \qquad \beta_s = N - 1$$

$$a_0 = B_{n+2} K \Big(\sum_{\alpha=1}^{n} p_\alpha' C_\alpha + p_x\Big)^N$$

$$b_0 = NKB_{n+2}\Big(\sum_{\alpha=1}^{n} p_\alpha' C_\alpha + p_x\Big)^{N-1} \Big(\sum_{\alpha=1}^{n} p_\alpha' D_\alpha + p_y\Big) + NKA_{n+2}\Big(\sum_{\alpha=1}^{n} p_\alpha' C_\alpha + p_x\Big)^N$$

The system (3.5) has to be subjected to transformations which make it possible to judge the stability of the motion just on the basis of the terms with $f_0(x)$ and $\phi_0(x)$. Omitting the discussion and derivations, which are analogous to the earlier ones, we shall write down immediately the necessary and sufficient conditions for the stability of the undisturbed motion of the system (1.1).

N has to be an odd number, and

$$KB_{n+2}\Big(\sum_{\alpha=1}^{n} p_\alpha' C_\alpha + p_x\Big)^N < 0$$

$$NKB_{n+2}\Big(\sum_{\alpha=1}^{n} p_\alpha' C_\alpha + p_x\Big)^{N-1}\Big(\sum_{\alpha=1}^{n} p_\alpha' D_\alpha + p_y\Big) + NKA_{n+2}\Big(\sum_{\alpha=1}^{n} p_\alpha' C_\alpha + p_x\Big)^N < 0 \qquad (3.6)$$

Just as in the preceding case, the requirement that N be odd can be reduced to the requirement that the characteristic of the control element be odd, and the conditions (3.6) make it possible to construct the region of admissible values of the parameters of the control system by starting with the stability of the undisturbed motion of the system (1.1).

4. Let us consider the case when the condition $\alpha_0 > \beta_0$ is violated, i.e. when $\beta_0 \geqslant \alpha_0$. It is obvious that in this case the problem on stability cannot be solved on the basis of the lowest-degree terms of the expansion of the functions $f_0(x)$ and $\phi_0(x)$, and that it is necessary to take into consideration higher-degree terms.

Suppose that $\alpha_0 < 0$, α_0 is an odd number, and $\beta_0 \geqslant (\alpha_0 - 1)/2$, or $\beta_0 \geqslant m$ where $\alpha_0 = 2m + 1$.

Without changing the problem on stability, let us introduce the transformation

$$x = r\cos\theta, \qquad y = -r^{m+1}\sin\theta \qquad (r > 0)$$

In place of (2.10) and (3.5) we then obtain the system

$$\frac{dr}{dt} = r^{m+1}R_1(\theta) + r^{m+2}R_2(\theta) + \ldots + r^{\alpha_0+N-m}\sum_{k=0}^{\infty} r^k R_k(\theta, y_1, \ldots, y_n)$$

$$\frac{d\theta}{dt} = r^m Q_0(\theta) + r^{m+1}Q_2(\theta) + \ldots + r^{\alpha_0+N-1-m}\sum_{k=0}^{\infty} r^k Q_k(\theta, y_1, \ldots, y_n) \qquad (4.1)$$

$$\frac{dy_s}{dt} = p_{s1}y_1 + \ldots + p_{sn}y_n + r^{\alpha_0+N}\sum_{k=0}^{\infty} r^k R_{sk}(\theta) + \sum_{k=0}^{\infty} r^k L_{sk}(\theta, y_1, \ldots, y_n)$$

$$(s = 1, \ldots, n)$$

Here

$$Q_0(\theta) = \frac{(m+1)\sin^2\theta + \cos^{2m+2}\theta}{1 + m\sin^2\theta}$$

$$R_k(\theta, 0, \ldots, 0) = 0, \qquad Q_k(\theta, 0, \ldots, 0) = 0, \qquad L_{sk}(\theta, 0, \ldots, 0) = 0$$

Following Liapunov [1], we define

$$g_1 = \int_0^{2\pi} \frac{R_1}{Q_0}\, d\theta$$

We next introduce the notation

$$G(\theta) = \int_0^{\theta} \left(g_1 - \frac{R_1}{Q_0}\right) d\theta$$

We note that $G(\theta)$ is a bounded periodic function of θ of period 2π.

Assuming that $g_1 \neq 0$, we introduce the substitution

$$\rho = re^{G(\theta)} \qquad (\rho > 0)$$

We now pass from the system (4.1) to the following system:

$$\frac{d\rho}{dt} = \rho^{m+1} g_1 P_1(\theta) + \rho^{m+2} P_2(\theta) + \ldots + \rho^{\alpha_0+N-m} \sum_{k=0}^{\infty} \rho^k P_k(\theta, y_1, \ldots, y_n) \quad (4.2)$$

$$\frac{d\theta}{dt} = \rho^m F_0(\theta) + \rho^{m+1} F_1(\theta) + \ldots + \rho^{\alpha_0+N-1-m} \sum_{k=0}^{\infty} \rho^k F_k(\theta, y_1, \ldots, y_n)$$

$$\frac{dy_s}{dt} = p_{s1} y_1 + \ldots + p_{sn} y_n + \rho^{\alpha_0+N} \sum_{k=0}^{\infty} \rho^k M_{sk}(\theta) + \sum_{k=0}^{\infty} \rho^k W_{sk}(\theta, y_1, \ldots, y_n)$$

$$(s = 1, \ldots, n)$$

We take Liapunov's function in the form

$$V = \rho + V_1(y_1, \ldots, y_n) \quad (4.3)$$

where

$$\sum_{s=1}^{n} \frac{\partial V_1}{\partial y_s} (p_{s1} y_1 + \ldots + p_{sn} y_n) = g_1 (y_1^2 + \ldots + y_n^2)$$

It is clear that if $g_1 > 0$, the form V_1 will be negative-definite, and if $g_1 < 0$, the form V_1 will be positive-definite.

Taking into account (4.2), we obtain

$$\frac{dV}{dt} = g_1 Q_0(\theta)\, e^{-mG(\theta)} \rho^{m+1} + g_1 (y_1^2 + \ldots + y_n^2) + \rho^{m+2} P_2(\theta) + \ldots$$

$$\ldots + \rho^{\alpha_0+N-m} \sum_{k=0}^{\infty} \rho^k P_k(\theta, y_1, \ldots, y_n) +$$

$$+ \sum_{s=1}^{n} \frac{\partial V_1}{\partial y_s} \left[\rho^{\alpha_0+N} \sum_{k=0}^{\infty} \rho^k M_{sk}(\theta) + \sum_{k=0}^{\infty} \rho^k W_{sk}(\theta, y_1, \ldots, y_n)\right]$$

Analysing the expression for dV/dt, we find that the sign of dV/dt is determined by the sign of g_1 for small enough values of ρ, $y_1, \ldots, y_n$.

If $g_1 < 0$, we have stability of motion; if $g_1 > 0$, we have instability of motion.

Determining $R_1(\theta)$ and $Q_0(\theta)$ for the given system, we find that a condition of stability of the undisturbed motion of the systems (2.10) and (3.5), with $a_0 < 0$, $\beta_0 \geqslant (a_0 - 1)/2$, is given by the inequality

$$\int_0^{2\pi} \frac{\sin\theta\cos\theta + a_0\cos^N\theta\sin\theta}{\frac{1}{2}(N+1)\sin^2\theta - a_0\cos^{N+1}\theta}\, d\theta > 0 \qquad (4.4)$$

The number N has to be odd. The condition (4.4) connects the parameters of the controlling element with the parameters of the object of control.

In case $g_1 = 0$, one has to pass to the next higher element, define

$$g_2 = \int_0^{2\pi} \frac{P_2}{F_0}\, d\theta$$

and carry out the investigation from here on as before. If, however, g_2 is also zero, then one has to find the first number g_k distinct from zero. If it is impossible to find such a number k that $g_k \neq 0$, while $g_1 = \ldots = g_{k-1} = 0$, then the problem on the stability of the given system remains unsolved.

5. Let us consider the case when the right-hand side terms of the first $n + 1$ equations of the system (1.1) contain nonlinear terms.

Suppose that in place of the system (1.1) we are given the system

$$\frac{dx_k}{dt} = \sum_{\alpha=1}^{n+1} b_{k\alpha}x_\alpha + n_k x_{n+2} + \Phi_k(x_1, \ldots, x_{n+2}) \qquad (k = 1, \ldots, n+1)$$

$$\frac{dx_{n+2}}{dt} = f(\sigma), \qquad \sigma = \sum_{\alpha=1}^{n+1} p_\alpha x_\alpha + p_{n+2}x_{n+2} \qquad (5.1)$$

The variable coefficients have here the same meaning as before; the function $f(\sigma)$ has the form (1.2), and the expansion of the function Φ_k in powers of its variables contains no terms of degree less than two; the roots of the characteristic equation of the system (5.1) are assumed to have the properties specified earlier.

Let

$$\Phi_k = \sum_{\alpha=1}^{n+2} d_{k\alpha} x_\alpha^2 + \sum_{\alpha=1}^{n+2} m_{k\alpha} x_\alpha^3 \qquad (k = 1, \ldots, n+1)$$

Applying the transformations considered above to the system (5.1), we obtain a system of the type (3.5) where

$$\sigma = \sum_{\alpha=1}^{n} p_\alpha' x_\alpha + \left(\sum_{\alpha=1}^{n} p_\alpha' C_\alpha + p_x\right) x + \left(\sum_{\alpha=1}^{n} p_\alpha' D_\alpha + p_y\right) y$$

$$Y = \sum_{k=1}^{n+1} B_k \Phi_k + \sum_{k=1}^{n+1} A_k \frac{d\Phi_k}{dt} + B_{n+2} f(\sigma) + A_{n+2} \frac{df(\sigma)}{dt}$$

$$X_s = -D_s Y + \Phi_s \qquad (s = 1, \ldots, n)$$

For the functions Y and X_s we again use the representations (2.4) and (2.5).

For the system under consideration we find that $\alpha_0 = 2$, $\beta_0 = 1$, and that the quantities a_0, b_0, a_1, and b_1 depend on $d_{k\alpha}$ and $m_{k\alpha}$, while

$$a_0 = a_0(d_{k\alpha}), \qquad b_0 = b_0(d_{k\alpha}), \qquad a_1 = a_1(d_{k\alpha}, m_{k\alpha}), \qquad b_1 = b_1(d_{k\alpha}, m_{k\alpha})$$

If $a_0 \neq 0$ and $b_0 \neq 0$, then the lowest-degree term of the expansion of the function $f_0(x)$ has an even degree, equal to two. In this case we have instability of motion.

For stability of motion it is necessary to make sure that the degree α_0 of the lowest-degree term of the function $f_0(x)$ be odd, and that the degree β_0 of the lowest-degree term of the expansion of $\phi_0(x)$ be even, while at the same time $\beta_0 > \alpha_0$.

Taking into account the results obtained earlier, we obtain the following necessary and sufficient conditions for the stability of the undisturbed motion of the system (5.1):

$$a_0(d_{k\alpha}) = 0, \qquad b_0(d_{k\alpha}) = 0 \tag{5.2}$$

$$a_1(d_{k\alpha}, m_{k\alpha}) < 0, \qquad b_1(d_{k\alpha}, m_{k\alpha}) < 0 \tag{5.3}$$

We call attention to the fact that it is impossible to fulfill the condition (5.2) when $d_{k\alpha} \neq 0$, and $N > 2$ in Expression (1.2). Hence, in order to guarantee stability in this case it is necessary to have $N = 2$ in Expression (1.2).

The conditions (5.3) and (5.2) connect the parameters of the controlling element with the coefficients of the equations of the object of control and among them with the coefficients of the nonlinear terms.

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